Introducing the Lagrangian and applying Euler's equation to it

```
In[]:= << VariationalMethods`

In[]:= L[t_] := 1/2 (m + M) D[r[t], t]^2 + 1/2 m*r[t]^2*D[θ[t], t]^2 + g*r[t]*(m*Cos[θ[t]] - M)
```

Applying Euler's equations for both r and θ yields:

```
In[]:= EulerEquations[L[t], r[t], t] // TraditionalForm
       Expand@EulerEquations[L[t], θ[t], t] // TraditionalForm
```

Out[]//TraditionalForm=

$$g\,m\cos(\theta(t)) - g\,M - (m+M)\,r''(t) + m\,r(t)\,\theta'(t)^2 = 0$$

Out[]//TraditionalForm=

$$-g\,m\,r(t)\sin(\theta(t)) - 2\,m\,r(t)\,r'(t)\,\theta'(t) - m\,r(t)^2\,\theta''(t) = 0$$

Simplifying the equations by introducing the mass ratio M/m = μ, where M is for the vertically moving ball and m is for the swinging one, and then eliminate the mass from θ equation :

```
In[]:= Solve[{EulerEquations[L[t], r[t], t], μ == M/m}, r''[t], {M, m}][[1]][[1]] /. Rule → Equal
       Solve[Expand@EulerEquations[L[t], θ[t], t], θ''[t]][[1]][[1]] /. Rule → Equal
```

$$\text{Out[]= } r''[t] == \frac{-g\mu + g\,\mathrm{Cos}[\theta[t]] + r[t]\,\theta'[t]^2}{1+\mu}$$

$$\text{Out[]= } \theta''[t] == \frac{-g\,\mathrm{Sin}[\theta[t]] - 2\,r'[t]\,\theta'[t]}{r[t]}$$

Solving the differential equation for multiple set of initial conditions with g=9.8 :

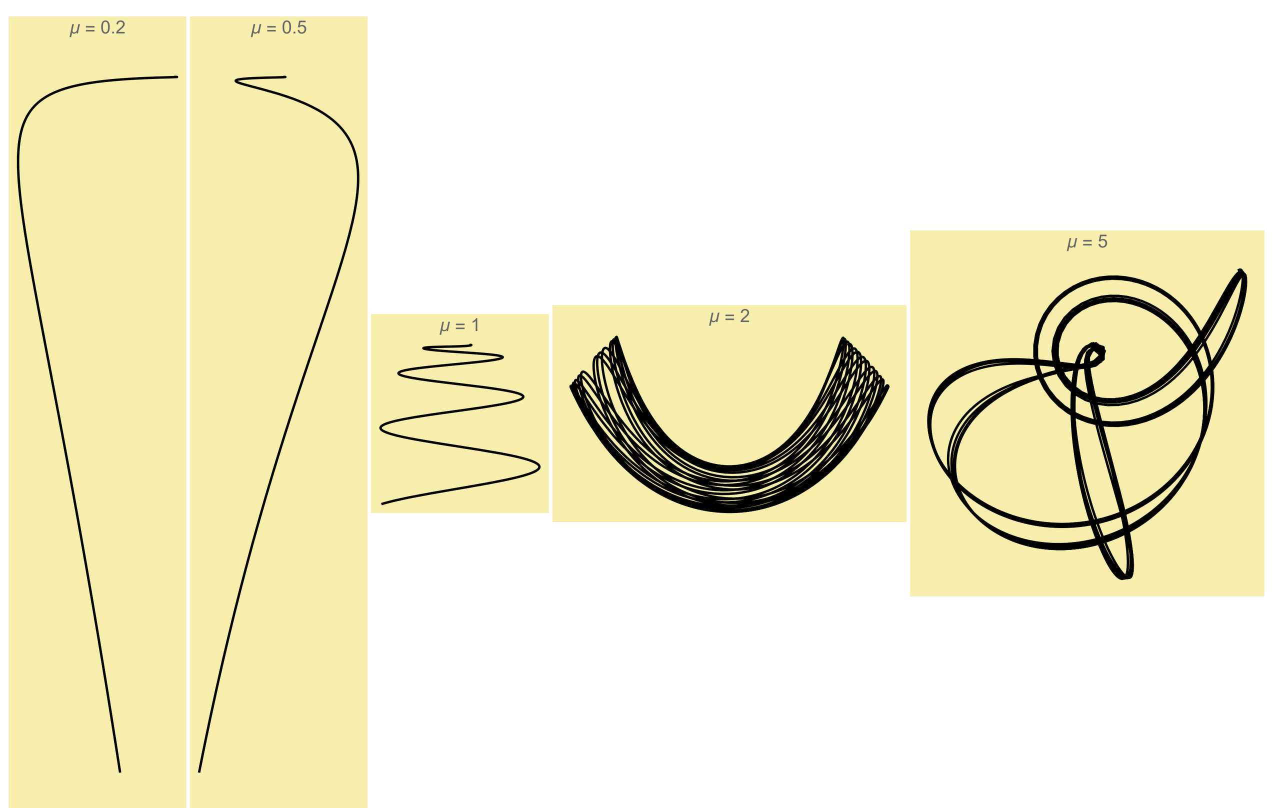
```
In[]:= g = 9.8;

In[]:= Dr[t_] := NDSolve[{r''[t] == 1/(1 + μ) (-g μ + g Cos[θ[t]] + r[t] θ'[t]^2), θ''[t] == (-g Sin[θ[t]] - 2 r'[t] θ'[t])/r[t], r'[0] == b, θ'[0] == d, r[0] == a, θ[0] == c}, {r[t], θ[t]}, {t, 0, 100}]
```

Plotting the the results in Polar coordinates

The first set of initial conditions is $r_0 = 1.5,\ \dot{r}_0 = 0.6,\ \theta_0 = \frac{2\pi}{3},\ \dot{\theta}_0 = 0$, each plot is labeled with the used μ

```
In[]:= a = 1.5; b = 0.6; c = 2π/3; d = 0;
       μ = 0.2;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]

       g1 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 50}, PlotStyle → Black, Background → RGBColor[0.97, 0.93, 0.68]], PlotLabel → HoldForm["μ = 0.2"],
         LabelStyle → {Thick}, ImageSize → Scaled[0.1], Axes → False, LabelStyle → Opacity[0]];
       μ = 0.5;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]

       g2 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 50}, PlotStyle → Black, Background → RGBColor[0.97, 0.93, 0.68]], PlotLabel → HoldForm["μ = 0.5"],
         LabelStyle → {Thick}, ImageSize → Scaled[0.1], Axes → False, LabelStyle → Opacity[0]];
       μ = 1;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]

       g3 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 50}, PlotStyle → Black, Background → RGBColor[0.97, 0.93, 0.68], AspectRatio → 1], PlotLabel → HoldForm["μ = 1"],
         LabelStyle → {Thick}, ImageSize → Scaled[0.1], Axes → False, LabelStyle → Opacity[0]];
       μ = 2;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]

       g4 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 30}, PlotStyle → Black, Background → RGBColor[0.97, 0.93, 0.68]], PlotLabel → HoldForm["μ = 2"],
         LabelStyle → {Thick}, ImageSize → Scaled[0.2], Axes → False, LabelStyle → Opacity[0]];
       μ = 5;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]

       g5 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 30}, PlotStyle → Black, PlotRange → All, Background → RGBColor[0.97, 0.93, 0.68]], PlotLabel → HoldForm["μ = 5"],
         LabelStyle → {Thick}, ImageSize → Scaled[0.2], Axes → False, LabelStyle → Opacity[0]];
       g1 g2 g3 g4 g5
```

Out[]=



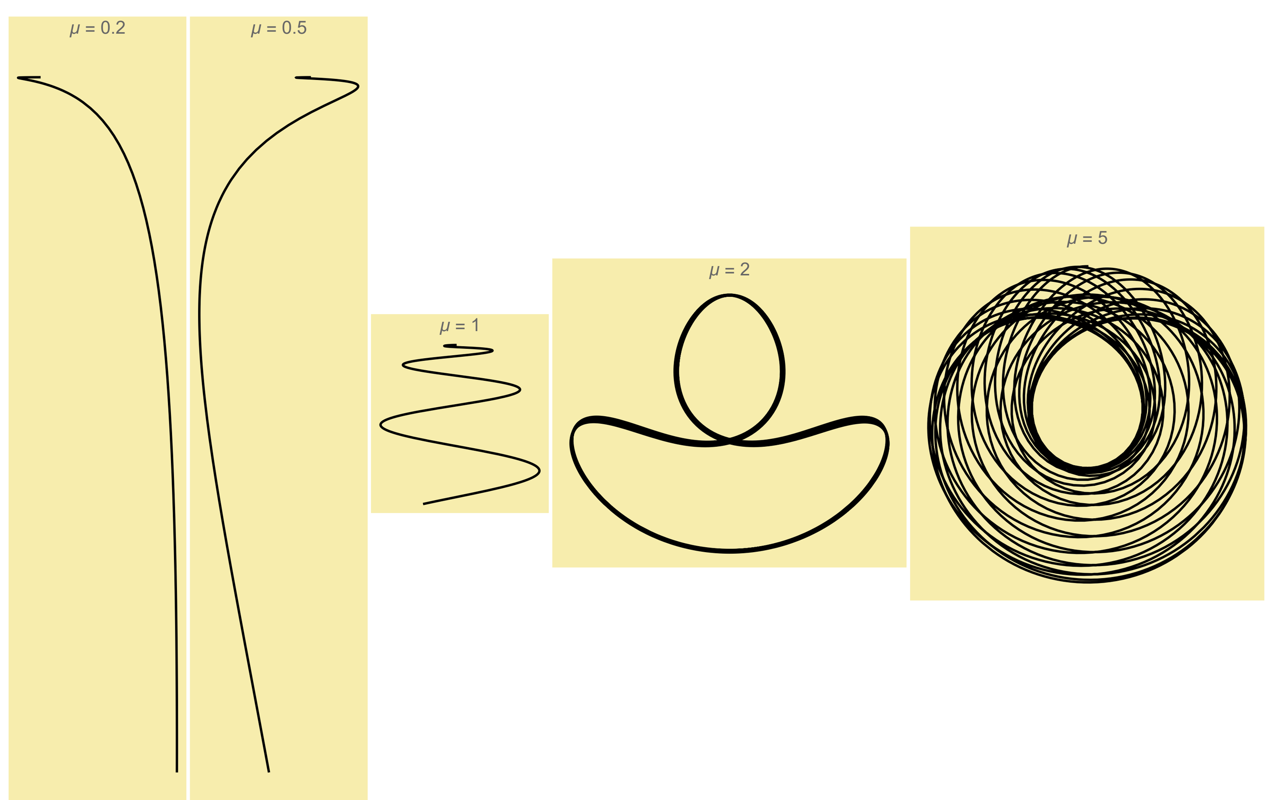
In[]:=

The Second set of initial conditions is $r_0 = 2,\ \dot{r}_0 = 0,\ \theta_0 = \pi,\ \dot{\theta}_0 = 1.5$, each plot is labeled with the used μ

In[]:=

```
In[]:= a = 2; b = 0; c = Pi; d = 1.5;
       μ = 0.2;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]

       h1 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 50}, PlotStyle → Black, Background → RGBColor[0.97, 0.93, 0.68]], PlotLabel → HoldForm["μ = 0.2"],
         LabelStyle → {Thick}, ImageSize → Scaled[0.1], Axes → False, LabelStyle → Opacity[0]];
       μ = 0.5;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]

       h2 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 50}, PlotStyle → Black, Background → RGBColor[0.97, 0.93, 0.68]], PlotLabel → HoldForm["μ = 0.5"],
         LabelStyle → {Thick}, ImageSize → Scaled[0.1], Axes → False, LabelStyle → Opacity[0]];
       μ = 1;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]

       h3 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 50}, PlotStyle → Black, Background → RGBColor[0.97, 0.93, 0.68], AspectRatio → 1], PlotLabel → HoldForm["μ = 1"],
         LabelStyle → {Thick}, ImageSize → Scaled[0.1], Axes → False, LabelStyle → Opacity[0]];
       μ = 2;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]

       h4 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 30}, PlotStyle → Black, Background → RGBColor[0.97, 0.93, 0.68]], PlotLabel → HoldForm["μ = 2"],
         LabelStyle → {Thick}, ImageSize → Scaled[0.2], Axes → False, LabelStyle → Opacity[0]];
       μ = 5;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]

       h5 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 30}, PlotStyle → Black, PlotRange → All, Background → RGBColor[0.97, 0.93, 0.68]], PlotLabel → HoldForm["μ = 5"],
         LabelStyle → {Thick}, ImageSize → Scaled[0.2], Axes → False, LabelStyle → Opacity[0]];
       h1 h2 h3 h4 h5
```

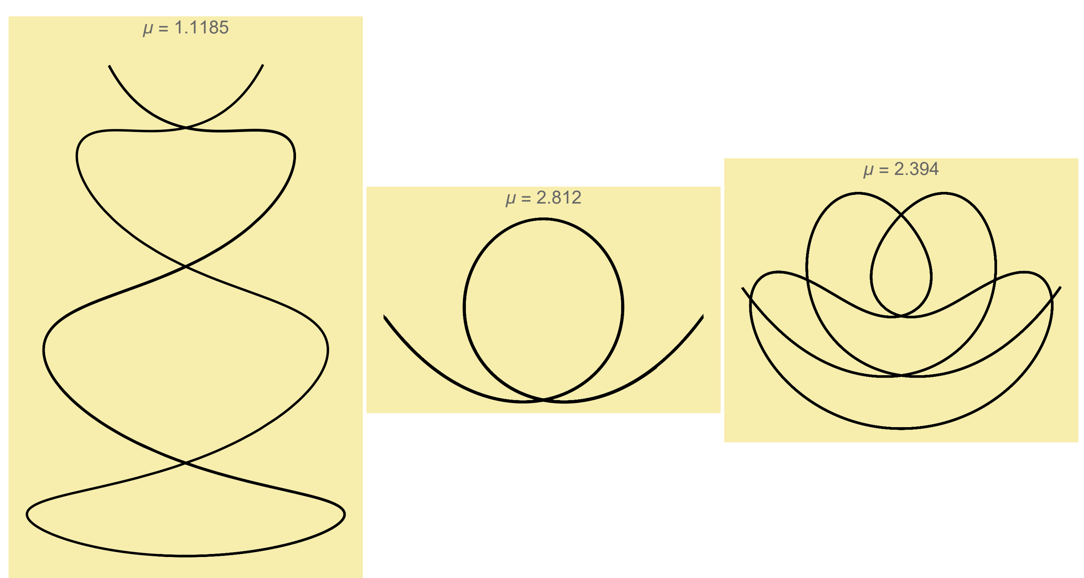
Out[]=



I will now reproduce some of the plots generated in Figure 4 from the paper Professor Bahlouli supplied where the initial conditions are $r_0 = 1,\ \dot{r}_0 = 0,\ \theta_0 = \frac{\pi}{2},\ \dot{\theta}_0 = 0$:

```
In[]:= a = 1; b = 0; c = Pi/2; d = 0;
       μ = 1.1185;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]
       q1 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 6 Pi}, PlotStyle → Black, Background → RGBColor[0.97`, 0.93`, 0.68`], AspectRatio → Automatic],
         PlotLabel → HoldForm["μ = 1.1185"], LabelStyle → {Thick}, ImageSize → Scaled[0.2], Axes → False, LabelStyle → Opacity[0]];
       μ = 2.394;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]
       q2 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 6 Pi}, PlotStyle → Black, Background → RGBColor[0.97`, 0.93`, 0.68`], AspectRatio → Automatic],
         PlotLabel → HoldForm["μ = 2.394"], LabelStyle → {Thick}, ImageSize → Scaled[0.2], Axes → False, LabelStyle → Opacity[0]];
       μ = 2.812;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]
       q3 = Show[ParametricPlot[Pr[t][[1]], {t, 0, 6 Pi}, PlotStyle → Black, Background → RGBColor[0.97`, 0.93`, 0.68`], AspectRatio → Automatic],
         PlotLabel → HoldForm["μ = 2.812"], LabelStyle → {Thick}, ImageSize → Scaled[0.2], Axes → False, LabelStyle → Opacity[0]];
       q1 q2 q3
```

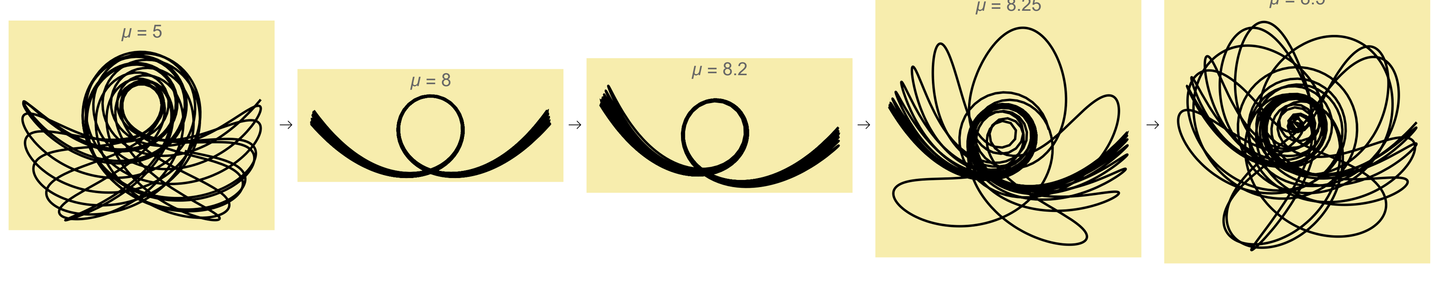
Out[]=



For the same set of initial conditions $r_0 = 1,\ \dot{r}_0 = 0,\ \theta_0 = \frac{\pi}{2},\ \dot{\theta}_0 = 0$, lets see at which value of μ the trajectory takes a chaotic behavior :

```
In[]:= a = 1; b = 0; c = Pi/2; d = 0;
       μ = 5;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]
       aa = Show[ParametricPlot[Pr[t][[1]], {t, 0, 6 Pi}, PlotStyle → Black, Background → RGBColor[0.97`, 0.93`, 0.68`], AspectRatio → Automatic],
         PlotLabel → HoldForm["μ = 5"], LabelStyle → {Thick}, ImageSize → Scaled[0.15], Axes → False, LabelStyle → Opacity[0]];
       μ = 8;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]
       bb = Show[ParametricPlot[Pr[t][[1]], {t, 0, 6 Pi}, PlotStyle → Black, Background → RGBColor[0.97`, 0.93`, 0.68`], AspectRatio → Automatic],
         PlotLabel → HoldForm["μ = 8"], LabelStyle → {Thick}, ImageSize → Scaled[0.15], Axes → False, LabelStyle → Opacity[0]];
       μ = 8.2;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]
       cc = Show[ParametricPlot[Pr[t][[1]], {t, 0, 6 Pi}, PlotStyle → Black, Background → RGBColor[0.97`, 0.93`, 0.68`], AspectRatio → Automatic],
         PlotLabel → HoldForm["μ = 8.2"], LabelStyle → {Thick}, ImageSize → Scaled[0.15], Axes → False, LabelStyle → Opacity[0]];
       μ = 8.25;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]
       dd = Show[ParametricPlot[Pr[t][[1]], {t, 0, 6 Pi}, PlotStyle → Black, Background → RGBColor[0.97`, 0.93`, 0.68`], AspectRatio → Automatic],
         PlotLabel → HoldForm["μ = 8.25"], LabelStyle → {Thick}, ImageSize → Scaled[0.15], Axes → False, LabelStyle → Opacity[0]];
       μ = 8.5;
       Pr[t_] := Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. Dr[t]]
       ee = Show[ParametricPlot[Pr[t][[1]], {t, 0, 6 Pi}, PlotStyle → Black, Background → RGBColor[0.97`, 0.93`, 0.68`], AspectRatio → Automatic],
         PlotLabel → HoldForm["μ = 8.5"], LabelStyle → {Thick}, ImageSize → Scaled[0.15], Axes → False, LabelStyle → Opacity[0]];
       aa → bb → cc → dd → ee
```

Out[]=



Some Observations:

We can observe from the First two sets of plots that when μ is one or less, which means the swinging ball is heavier than the vertical one, the ball will fall indefinity due to gravity. and once it exceeds 1, the trajectory will take a behavior that will not cause the swinging ball to fall indefinity.

The third set of plots shows that my work is precise and matches the scientific paper when the same initial conditions is used.

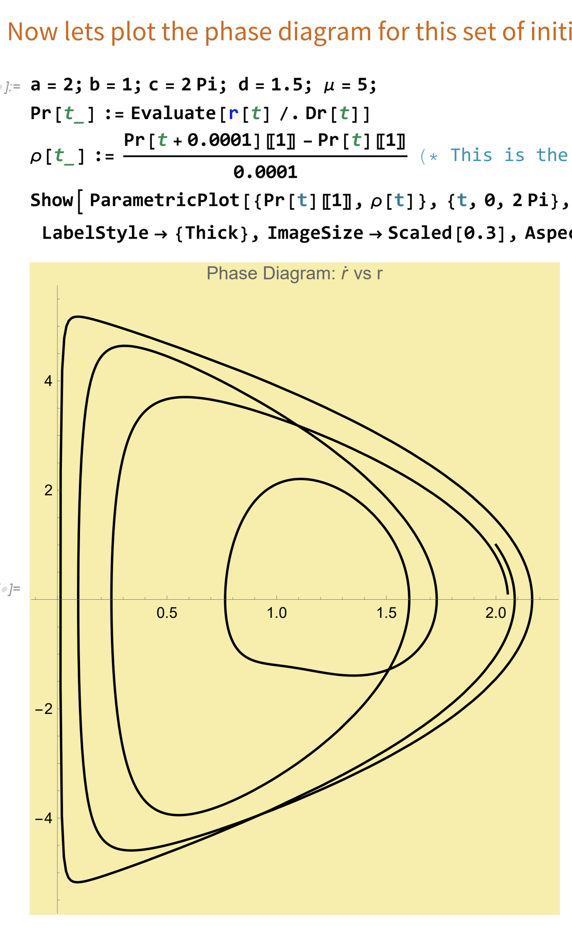
The final sets of plots shows that the trajectory of the swinging ball will dramatically be chaotic when μ exceeds 8.5

Phase diagram & interactive Plot

Now lets plot the phase diagram for this set of initial conditions $r_0 = 2,\ \dot{r}_0 = 1,\ \theta_0 = 2\pi,\ \dot{\theta}_0 = 1.5$ and this value of $\mu = 5$:

```
In[]:= a = 2; b = 1; c = 2 Pi; d = 1.5; μ = 5;
       Pr[t_] := Evaluate[r[t] /. Dr[t]]
       ρ[t_] := (Pr[t + 0.0001][[1]] - Pr[t][[1]])/0.0001 (* This is the numerical derivative of r, i.e. ṙ*)
       Show[ParametricPlot[{Pr[t][[1]], ρ[t]}, {t, 0, 2 Pi}, PlotStyle → Black, Background → RGBColor[0.97`, 0.93`, 0.68`]], PlotLabel → HoldForm["Phase Diagram: ṙ vs r"],
        LabelStyle → {Thick}, ImageSize → Scaled[0.3], AspectRatio → Full]
```

Out[]=

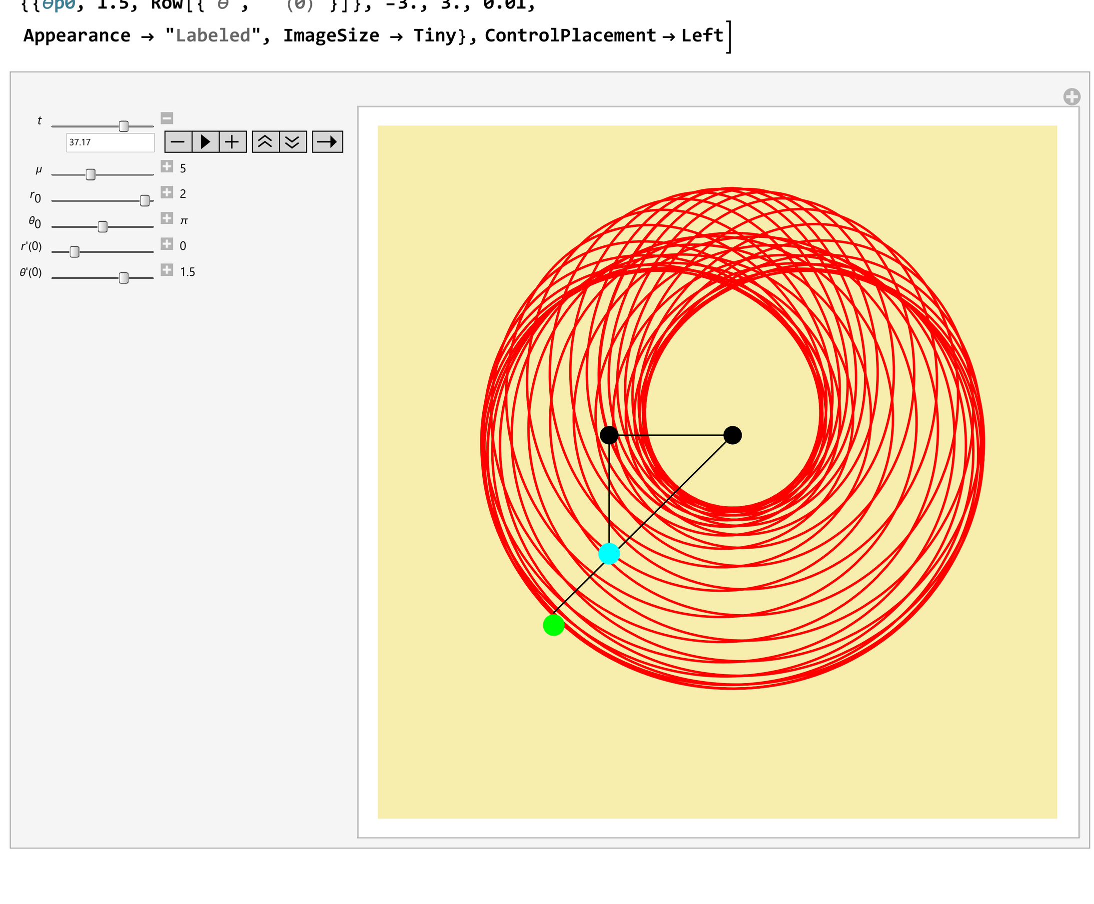


In[]:=

Here is the interaction plot, obtained from https://demonstrations.wolfram.com/SwingingAtwoodsMachine/ , with small modifications:

```
In[]:= Manipulate[
        With[
         {s = NDSolve[{r''[t] == 1/(1 + μ) (r[t] θ'[t]^2 - 9.8 (μ - Cos[θ[t]])), θ''[t] == -(2 r'[t] θ'[t])/r[t] - (9.8 Sin[θ[t]])/r[t], r'[0] == rp0, θ'[0] == θp0, r[0] == r0, θ[0] == θ0},
            {r, θ}, {t, 0, tf}, MaxSteps → ∞]}, ParametricPlot[Evaluate[r[t] {Cos[θ[t] - π/2], Sin[θ[t] - π/2]} /. s], {t, 0, tf},
          Background → RGBColor[0.97`, 0.93`, 0.68`], Axes → False, PlotRange → {{-2.5, 2.5}, {-3.1, 2.5}}, ImageSize → 450, PlotStyle → Red,
          ImagePadding → {{30, 10}, {Automatic, Automatic}},
          Epilog → {Line[{{-1, 0}, {0, 0}}], Line[{{-1., 0}, {-1., -3 + r[tf]} /. First[s]}], Disk[{-1, 0}, .075], Disk[{0, 0}, .075], PointSize[.035],
            Rotate[{Line[{{0, 0}, {r[tf] Cos[θ[tf]], r[tf] Sin[θ[tf]]} /. First[s]}], Green, Point[{r[tf] Cos[θ[tf]] + .1, r[tf] Sin[θ[tf]]} /. First[s]]}, -π/2, {0, 0}],
            Cyan, Point[{-1., -3 + r[tf]} /. First[s]]}, PlotPoints → 100]],
        {{tf, 0.1, Style["t", Italic]}, 0.1, 50, 0.01, AnimationRate → 1,
        Appearance → "Open", ImageSize → Tiny}, {{μ, 5, "μ"}, 1.26, 12, 0.01,
        Appearance → "Labeled", ImageSize → Tiny},
        {{r0, 2, Subscript[Style["r", Italic], 0]}, 0.2, 2, 0.01, Appearance → "Labeled",
        ImageSize → Tiny}, {{θ0, Pi, Subscript["θ", 0]}, 0., 6.28, 0.01,
        Appearance → "Labeled", ImageSize → Tiny},
        {{rp0, 0, Row[{Style["r", Italic], "'(0)"}]}, -0.5, 2.62, 0.01,
        Appearance → "Labeled", ImageSize → Tiny},
        {{θp0, 1.5, Row[{"θ", "'(0)"}]}, -3., 3., 0.01,
        Appearance → "Labeled", ImageSize → Tiny}, ControlPlacement → Left]
```

Out[]=



Ibraheem Al-Yousef, PHYS300