Q 1:

$$\psi(x) = \frac{1}{\sqrt{L}}e^{ikx}$$

a):

$$\langle x\rangle = \langle\psi|x|\psi\rangle = \frac{1}{L}\int_0^L e^{-ikx}xe^{ikx}dx = \frac{1}{L}\int_0^L x\ dx\ = \frac{L}{2}$$

We can interpret this as "The most probable position is located at L/2"

b):

$$\langle p_x\rangle = \langle\psi|p_x|\psi\rangle = \frac{\hbar}{iL}\int_0^L e^{-ikx}\frac{d}{dx}e^{ikx}dx = \frac{\hbar ki}{iL}\int_0^L dx = \hbar k$$

c):

$$\langle p_x^2\rangle = \langle\psi|p_x^2|\psi\rangle = \frac{-\hbar^2}{L}\int_0^L e^{-ikx}\frac{d^2}{dx^2}e^{ikx}dx = \frac{\hbar^2k^2}{L}\int_0^L dx = \hbar^2k^2$$

d):

$$\frac{\langle p_x^2\rangle}{2m_0} = \frac{\hbar^2k^2}{2m_0}; \quad knowing\ k^2 = \frac{2m_0(E-U_0)}{\hbar^2}$$
$$= \frac{\hbar^2 2m_0(E-U_0)}{2m_0\hbar^2} = E - U_0\checkmark$$

Q 2:

$$\psi_1(x) = \sqrt{\frac{2}{a}}\sin(\frac{\pi x}{a})$$

a):

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi; \qquad k^2 \equiv \frac{2mE}{\hbar^2}$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \implies \psi(x) = A\sin(kx) + B\cos(kx)$$

After applying $\psi(0) = \psi(L) = 0$ & $\psi'(0) = \psi'(L) = 0$:

$$\psi_n(x) = A\sin\frac{n\pi x}{a}$$

$$\langle\psi_n|\psi_m\rangle = \delta_{nm} \implies A = \sqrt{\frac{2}{a}}$$

$$\therefore \psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a}) \qquad \square$$

b):

$$\langle x \rangle = \langle \psi_1 | x | \psi_1 \rangle = \frac{2}{a} \int_0^a x \sin^2(\frac{\pi x}{a}) dx = \frac{1}{a} \int_0^a x - x \cos(\frac{2\pi x}{a}) dx = \frac{1}{2} a$$

We can interpret this as "The most probable position is located at a/2"

c):

$$\langle p_x \rangle = \langle \psi_1 | p_x | \psi_1 \rangle = \frac{2\hbar}{ia} \int_0^a \sin(\frac{\pi x}{a}) \frac{d}{dx} \sin(\frac{\pi x}{a}) dx = \frac{2\hbar\pi}{a^2 i} \int_0^a \sin(\frac{\pi x}{a}) \cos(\frac{\pi x}{a}) dx = 0$$

This can be interpreted as this particle is a wave-like and has zero momentum.

d):

$$\langle p_x^2 \rangle = \langle \psi_1 | p_x^2 | \psi_1 \rangle = -\frac{2\hbar^2}{a} \int_0^a \sin(\frac{\pi x}{a}) \frac{d^2}{dx^2} \sin(\frac{\pi x}{a}) dx = \frac{2\pi^2\hbar^2}{a^3} \int_0^a \sin^2(\frac{\pi x}{a}) dx$$
$$= \frac{\pi^2\hbar^2}{a^3} \int_0^a 1 - \cos(\frac{2\pi x}{a}) dx = \frac{\pi^2\hbar^2}{a^2}$$

e):

$$\frac{\langle p_x^2 \rangle}{2m_0} = \frac{\pi^2\hbar^2}{2m_0 a^2} = E_1 \checkmark; \qquad E_n = \frac{n^2\pi^2\hbar^2}{2m_0 a^2}$$

Q 3:

$$E = 100\ keV$$
$$E = \frac{1}{2} m v^2 \implies v = \sqrt{\frac{2E}{m}}$$
$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{m}\sqrt{\frac{m}{2E}} = 3.88\ pm = 0.00388\ nm << 193\ nm$$

Q 4:

a):

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + U_0 \psi = E\psi$$
$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - U_0)\psi = 0$$
$$\frac{d^2\psi}{dx^2} - \alpha^2 \psi = 0; \quad since\ U_0 > E,\ \alpha = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$$

b):

$$E = \frac{3k_BT}{2} = 0.0387\ eV; \qquad U_0 = 3.38\ eV$$

$$\frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = 0.107\ nm$$

Q 5:

$$k_1 = \zeta\sqrt{E}; \quad k_2 = \zeta\sqrt{E - U_0}; \qquad \zeta = \frac{\sqrt{2m}}{\hbar}$$

$$\psi(0_+) = \psi(0_-) \implies 1 + r = t \tag{1}$$

$$\psi'(0_+) = \psi'(0_-) \implies k_1 - rk_1 = tk_2 \tag{2}$$

$$(1)\ in\ (2) \implies k_1 - rk_1 = k_2 + rk_2 \implies r = \frac{\zeta\sqrt{E} - \zeta\sqrt{E - U_0}}{\zeta\sqrt{E} + \zeta\sqrt{E - U_0}}$$

$$|r^2| = \frac{|2E - 2\sqrt{E^2 - EU_0} - U_0|}{|2E + 2\sqrt{E^2 - EU_0} - U_0|}$$

As the energy increases above U_0, the reflection coefficient will become smaller and smaller rapidly, as we can see in Fig. 1:

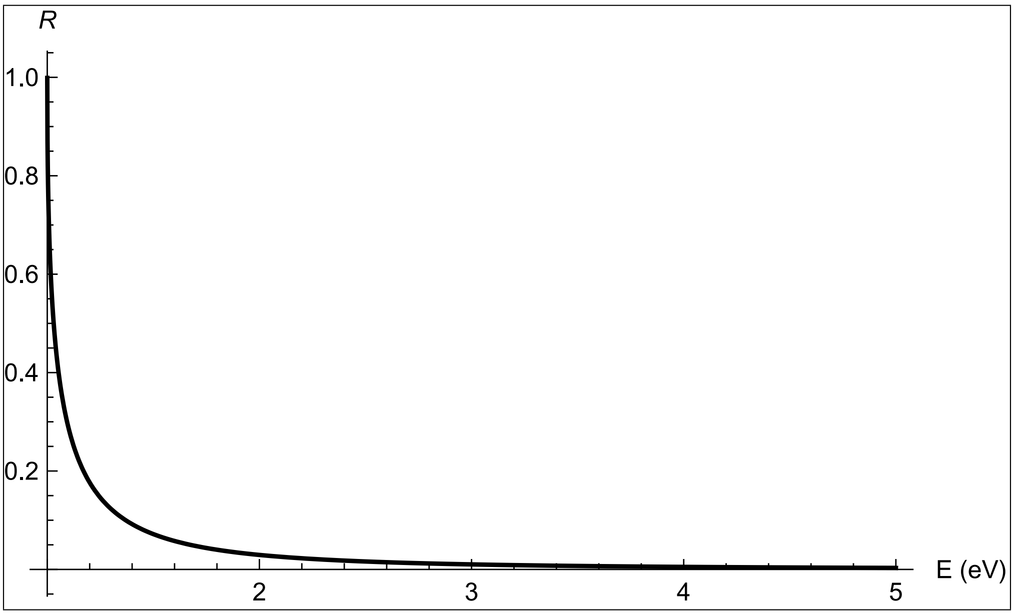

Figure 1: R Vs. E, for $U_0 = 1\ eV$

Q 6:

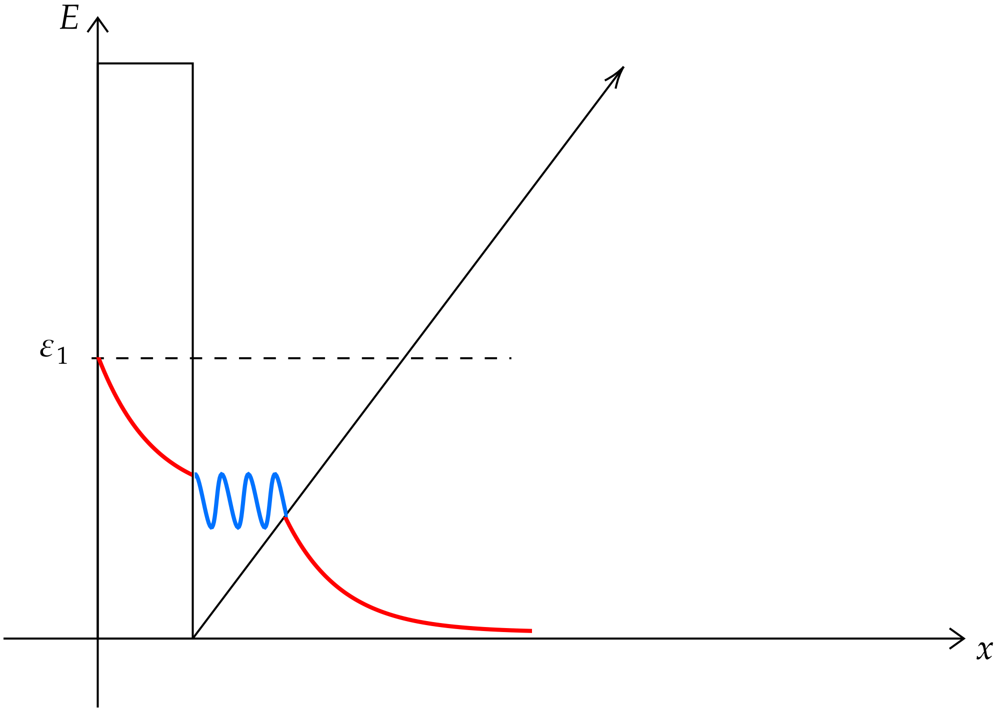

Figure 2: Sketch of $\psi(x)$

For this question, we only need to determine whether $\varepsilon_1 - V$ is positive or negative, is positive, then the wavefunction will be a traveling wave. If negative, the wavefunction will be a decaying wave.

Q 7:

a) Only one band is visible, so I set it to 3 eV and the energy band is $2.86 - 0.93 = 1.93$ eV
b) It increased by ≈ 0.5 eV
c) at 3 eV Band 1's width: 2.15, Band 2: 1.27
at 4 eV Band 1's width: 2.475, Band 2: 1.834
Hence, bandwidths became **wider**.