Q 1:

a):

$$P_A^\mu = P_B^\mu + P_C^\mu \implies P_B^\mu = P_A^\mu - P_C^\mu \implies P_B^2 = P_A^2 + P_C^2 - 2P_A^2 \cdot P_C^2$$

$$P_i^2 = m_i^2 c^2; \qquad E_i = m_i c^2; \qquad P_i^2 \cdot P_j^2 = \left(\frac{E_i E_j}{c^2} - \vec{p}_i \cdot \vec{p}_j \right)$$

$$Since\ \vec{p}_A = 0;\ Then\ m_B^2 c^2 = m_A^2 + m_C^2 - 2m_A m_C c^2$$

$$\therefore E_C = \frac{m_A^2 + m_C^2 - m_B^2}{2m_A} c^2; \qquad E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A} c^2$$

b):

$$Using\ |\vec{p}_i| = \sqrt{\frac{E_i^2 - (m_i c^2)^2}{c^2}}$$

$$|\vec{p}_B| = \sqrt{\frac{E_B^2 - (m_B c^2)^2}{c^2}} = \sqrt{\frac{m_A^4 + m_B^4 + m_C^4 + 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}{4m_A^2} c^2 - m_B^2 c^2}$$

$$|\vec{p}_B| = \frac{c}{2m_A} \sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2} = |\vec{p}_C|$$

c): If $m_A < (m_B + m_C)$, then there is no enough energy in particle A to produce B and C, (Assuming we are in particle A CM frame)

Q 2:

$$m_1 c \equiv \sqrt{r^2 + m_2^2 c^2} + \sqrt{r^2 + m_3^2 c^2}$$

$$m_1^2 c^2 = 2r^2 + m_2^2 c^2 + m_3^2 c^2 + 2\sqrt{r^2 + m_2^2 c^2}\sqrt{r^2 + m_3^2 c^2}$$

$$\left((m_1^2 - m_2^2 - m_3^2)\frac{c^2}{2} - r^2 \right)^2 = r^4 + r^2 m_2^2 c^2 + r^2 m_3^2 c^2 + m_2^2 m_3^2 c^4$$

$$(m_1^2 - m_2^2 - m_3^2)^2 \frac{c^4}{4} + r^4 - r^2 c^2 (m_1^2 - m_2^2 - m_3^2) = r^4 + r^2 c^2 (m_2^2 + m_3^2) + m_2^2 m_3^2 c^4$$

$$(m_1^2 - m_2^2 - m_3^2)^2 \frac{c^2}{4} = r^2 m_1^2 + m_2^2 m_3^2 c^2$$

$$r = \sqrt{\frac{(m_1^2 - m_2^2 - m_3^2)^2 c^2 - 4m_2^2 m_3^2}{4m_1^2}}$$

$$r = \frac{c}{2m_1} \sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}$$

Q 3:

$$\Gamma = \frac{S|\vec{p}|}{8\pi\hbar m_\pi^2 c}|\mathscr{M}|^2;\ n=3 \implies \mathscr{M} = \alpha m_\pi c$$

$$\Gamma = \frac{\alpha^2 c}{16\pi\hbar}|\vec{p}|; \qquad |\vec{p}| = E_\gamma/c = \frac{1}{2}m_\pi c$$

$$\tau = \frac{1}{\Gamma} = \frac{32\pi\hbar}{\alpha^2 c^2} = 9.2\times 10^{-18} \approx \frac{Experimental}{10}$$

Q 4 :

$$\sqrt{(P_1\cdot P_2)^2 - (m_1 m_2 c^2)^2} = ?$$

a) CM frame:

$$\vec{p}_1 = \vec{-}p_2 \implies P_1\cdot P_2 = \frac{E_1E_2}{c^2} - \vec{p}_1\cdot(-\vec{p}_2) = \frac{E_1E_2}{c^2} + \vec{p}_1^2$$

$$\sqrt{\left(\frac{E_1E_2}{c^2} + \vec{p}_1^2\right)^2 - (m_1m_2c^2)^2} = \sqrt{\frac{E_1^2E_2^2}{c^4} + \vec{p}_1^4 + 2\frac{E_1E_2}{c^2}\vec{p}_1^2 - m_1^2m_2^2c^4}$$

$$Knowing:\ m_i^2c^2 = \frac{E_i^2}{c^2} - \vec{p}_i^2; \qquad \vec{p}_1^2 = \vec{p}_2^2$$

$$\implies \sqrt{\frac{E_1^2E_2^2}{c^4} + \vec{p}_1^4 + 2\frac{E_1E_2}{c^2}\vec{p}_1^2 - (\frac{E_1^2}{c^2} - \vec{p}_1^2)(\frac{E_2^2}{c^2} - \vec{p}_1^2)}$$

$$\implies \sqrt{\frac{E_1^2E_2^2}{c^4} + \vec{p}_1^4 + 2\frac{E_1E_2}{c^2}\vec{p}_1^2 - \frac{E_1^2E_2^2}{c^4} - \vec{p}_1^4 + \frac{E_1^2}{c^2}\vec{p}_1^2 + \frac{E_2^2}{c^2}\vec{p}_1^2}$$

$$\therefore \sqrt{2\frac{E_1E_2}{c^2}\vec{p}_1^2 + \frac{E_1^2}{c^2}\vec{p}_1^2 + \frac{E_2^2}{c^2}\vec{p}_1^2} = \sqrt{\left(\frac{E_1}{c}\vec{p}_1 + \frac{E_2}{c}\vec{p}_1\right)^2} = \frac{E_1+E_2}{c}|\vec{p}_1|$$

$$\therefore \sqrt{(P_1\cdot P_2)^2 - (m_1m_2c^2)^2} = \frac{E_1+E_2}{c}|\vec{p}_1|$$

b) Lab frame, i.e. $\vec{p}_2 = 0$:

$$P_1\cdot P_2 = \frac{E_1 m_2 c^2}{c^2} \implies \sqrt{(P_1\cdot P_2)^2 - (m_1m_2c^2)^2} = \sqrt{(E_1m_2)^2 - (m_1m_2c^2)^2}$$

$$\implies \sqrt{E_1^2m_2^2 - m_1^2m_2^2c^4} = \sqrt{m_2^2(E_1^2 - m_1^2c^4)} = m_2c|\vec{p}_1|$$

$$\therefore \sqrt{(P_1\cdot P_2)^2 - (m_1m_2c^2)^2} = m_2c|\vec{p}_1|$$

Q 5:

$$\Gamma = \frac{S|\vec{p}|}{8\pi\hbar m_1^2 c}|\mathscr{M}|^2 \tag{1}$$

$$\Gamma = \frac{S}{8\pi\hbar m_1}\int_0^\infty |\mathscr{M}|^2 \frac{\delta\left(m_1c - \sqrt{r^2+m_2^2c^2} - \sqrt{r^2+m_3^2c^2}\right)}{\sqrt{r^2+m_2^2c^2}\sqrt{r^2+m_3^2c^2}} r^2 dr \tag{2}$$

By comparing Eq. (1) & (2):

$$\frac{|\vec{p}|}{m_1c}|\mathscr{M}|^2 = \int_0^\infty |\mathscr{M}|^2 \frac{\delta\left(m_1c - \sqrt{r^2+m_2^2c^2} - \sqrt{r^2+m_3^2c^2}\right)}{\sqrt{r^2+m_2^2c^2}\sqrt{r^2+m_3^2c^2}} r^2 dr \tag{3}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\vec{p}_1|}\int_0^\infty |\mathscr{M}|^2 \frac{\delta\left(\frac{E_1+E_2}{c} - \sqrt{r^2+m_2^2c^2} - \sqrt{r^2+m_3^2c^2}\right)}{\sqrt{r^2+m_2^2c^2}\sqrt{r^2+m_3^2c^2}} r^2 dr$$

Using Eq. (3) with $m_1c = \frac{E_1+E_2}{c}$. With $|\vec{p}_1| = |\vec{p}_i|$ & $|\vec{p}| = |\vec{p}_f|$ where $|\vec{p}_i|$ is the magnitude of one of the incoming momenta, and $|\vec{p}_f|$ is the magnitude of one of the outgoing momenta:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\vec{p}_i|}\frac{|\vec{p}_f|}{\frac{E_1+E_2}{c}}|\mathscr{M}|^2 = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathscr{M}|^2}{(E_1+E_2)^2}\frac{|\vec{p}_f|}{|\vec{p}_i|}$$

$$\therefore \frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathscr{M}|^2}{(E_1+E_2)^2}\frac{|\vec{p}_f|}{|\vec{p}_i|} \checkmark$$